A New Analytical Solution to the Multigroup Diffusion Equation in One Dimensional Plane Geometry

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+ One-group approximation
+ Multigroup diffusion equation
+ Second order ODEs
+ Homogeneous/particular solution
+ Eigenvectors/eigenvalues
+ Plane/curvilinear geometry
+ Transient solution

ABSTRACT

An analytical solution to the time-independent multigroup diffusion equation in heterogeneous plane media is presented. The solution features the simplicity of the one-group case which is rather remarkable given the generality of the multigroup diffusion equation considered. Beginning with the vector multigroup diffusion equation, the solution is based upon straightforward application of the mathematical principles associated with solving a set of second order ODEs. Once the homogeneous solution is known, the particular solution can be derived by the method of variation of parameters. In this way, the solution is formed in a recursive setting for which, with some effort, an explicit analytical expression can be derived. As a result, a new criticality relation emerges to yield the criticality constant given the slab thicknesses. Based on this analytical solution, it now becomes possible to construct an efficient multidimensional nodal method as well as a 1D transient solution using a numerical Laplace transform inversion. More importantly, however, is the educational principle that this new solution represents, i.e., Even though a problem has been solved in the past, new more efficient solutions should always be sought.

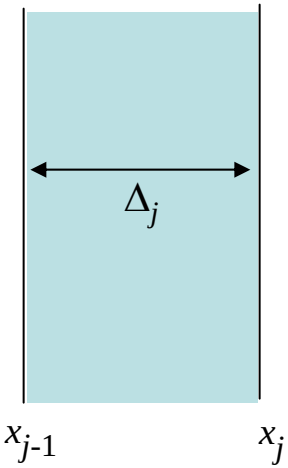
❒ Motivation:

+ One-group diffusion theory: Homogeneous medium

$$\left[D_j \frac{d^2}{dx^2} - \Sigma_{aj}\right]\phi_j(x) + \nu\Sigma_{fj}\phi_j(x) = -Q_j(x)$$

$$\left.\begin{aligned} \phi_{j-1} &\equiv \phi\left(x_{j-1}\right) \\ \phi_j &\equiv \phi\left(x_j\right) \end{aligned}\right\}$$

$$\Rightarrow \left[\frac{d^2}{dx^2} + B_j^2\right]\phi_j(x) = -q_j(x)$$



- Standard Solution:

$$\phi_j(x) = a_j e^{-B_j x} + b_j e^{B_j x} + \phi_{pj}(x)$$

$$\phi_{pj}^{-} \equiv \phi_p\left(x_{j-1}\right), \quad \phi_{pj}^{+} \equiv \phi_p\left(x_j\right)$$

$$a_j e^{-B_j x_{j-1}} + b_j e^{B_j x_{j-1}} = \phi_{j-1} - \phi_{pj}^{-}$$
$$a_j e^{-B_j x_j} + b_j e^{B_j x_j} = \phi_j - \phi_{pj}^{+}$$

$$\begin{bmatrix} a_j \\ b_j \end{bmatrix} = \boldsymbol{E}_j^{-1} \begin{bmatrix} \phi_{j-1} - \phi_{pj}^{-} \\ \phi_j - \phi_{pj}^{+} \end{bmatrix} \qquad \boldsymbol{E}_j \equiv \begin{bmatrix} e^{-B_j x_{j-1}} & e^{B_j x_{j-1}} \\ e^{-B_j x_j} & e^{B_j x_j} \end{bmatrix}$$

- Non-standard Solution:

$$\phi_j(x) = h_j^+(x)\left(\phi_j - \phi_{pj}^+\right) + h_j^-(x)\left(\phi_{j-1} - \phi_{pj}^-\right) + \phi_{pj}(x)$$

$$\left.\begin{aligned} h_j^+(x) &\equiv \left[\frac{\sin\left(B_j\left(x - x_{j-1}\right)\right)}{\sin\left(B_j\Delta_j\right)}\right] \\ h_j^-(x) &\equiv \left[\frac{\sin\left(B_j\left(x_j - x\right)\right)}{\sin\left(B_j\Delta_j\right)}\right] \end{aligned}\right\} \text{Basis Functions}$$

- Criticality: First Require $\phi_{pj}(x) \equiv 0$

Choose $x = x^*$, then

$$\phi_j(x^*) = h_j^+(x^*)\phi_j + h_j^-(x^*)\phi_{j-1}$$

$$\phi_j(x^*) = \sin\left(B_j\left(x^* - x_{j-1}\right)\right)\left[\frac{\phi_j}{\sin\left(B_j\Delta_j\right)}\right] + \sin\left(B_j\left(x_j - x^*\right)\right)\left[\frac{\phi_{j-1}}{\sin\left(B_j\Delta_j\right)}\right]$$

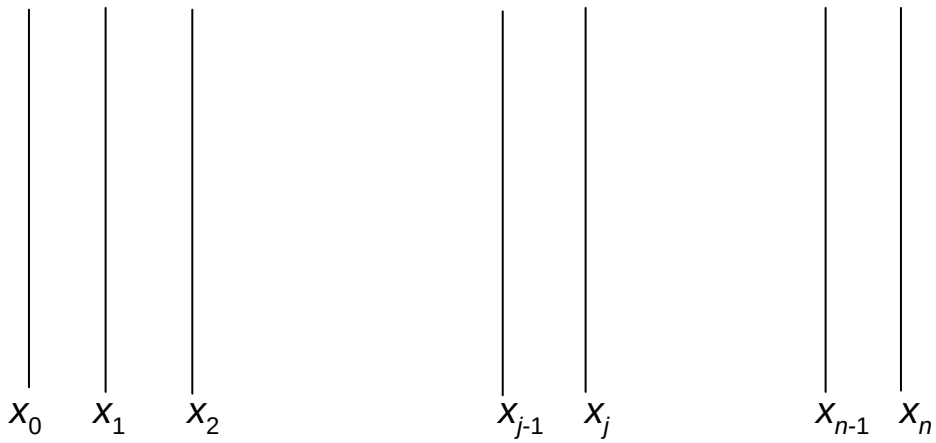
Also must require: $\phi_{j-1} = \phi_j \equiv 0$

Implies: $\phi_j(x^*) = 0?$ Not physically possible for criticality

Implies instead: $\sin\left(B_j\Delta_j\right) = 0 \quad \Rightarrow B_j(k_{eff}) = \dfrac{\pi}{\Delta_j}$

Implies: Denominator must vanish also!!

+ One-group diffusion theory: Heterogeneous medium



- Standard Solution:

$$\phi_j(x) = a_j e^{-B_j x} + b_j e^{B_j x} + \phi_{pj}(x)$$

$$-D_{j-1} \left.\frac{d\phi_{j-1}(x)}{dx}\right|_{x_{j-1}} = -D_j \left.\frac{d\phi_j(x)}{dx}\right|_{x_j} \qquad 2 \le j \le n-1$$

$$\Rightarrow \left\{ D_{j-1} a_{j-1} e^{-B_j x_{j-1}} + D_{j-1} b_{j-1} e^{B_j x_{j-1}} - D_j a_j e^{-B_j x_j} + D_j b_j e^{B_j x_j} = 0; \;\; j = 1, n \right\}$$ 2*n* unknowns

- Non-standard Solution:

3-term recurrence relation with *n*-2 unknowns

$$\alpha_j \phi_{j+1} + \beta_j \phi_j + \gamma_j \phi_{j-1} = f_j$$

+ Two-group diffusion theory: Core

$$\left[D_1 \frac{d^2}{dx^2} - \Sigma_1\right]\phi_1(x) + \chi_1 \nu \Sigma_{f1}\phi_1(x) + \chi_2 \nu \Sigma_{f2}\phi_2(x) = 0$$

$$\left[D_2 \frac{d^2}{dx^2} - \Sigma_2\right]\phi_2(x) + p\Sigma_{1,2}\phi_1(x) = 0$$

$$\left[\frac{d^2}{dx^2} + B^2\right]\Psi_k(x) = 0, \quad k = 1,2 \qquad \Rightarrow B^2 = \begin{cases} \mu^2 \\ -\nu^2 \end{cases}$$

$$\phi_1(x) = AX + CY \qquad X \equiv \sin(\mu x)$$

$$\phi_2(x) = S_1 AX + S_2 C \qquad Y \equiv \sinh(\nu x)$$

"Lamarsh's text

- Apply BCs for critical condition

+ Two-group diffusion theory: Core/Reflector

- Interfacial conditions/BCs give 4 simultaneous equations for the critical condition
- Complexity greatly increases with number of regions and groups

The Challenge:

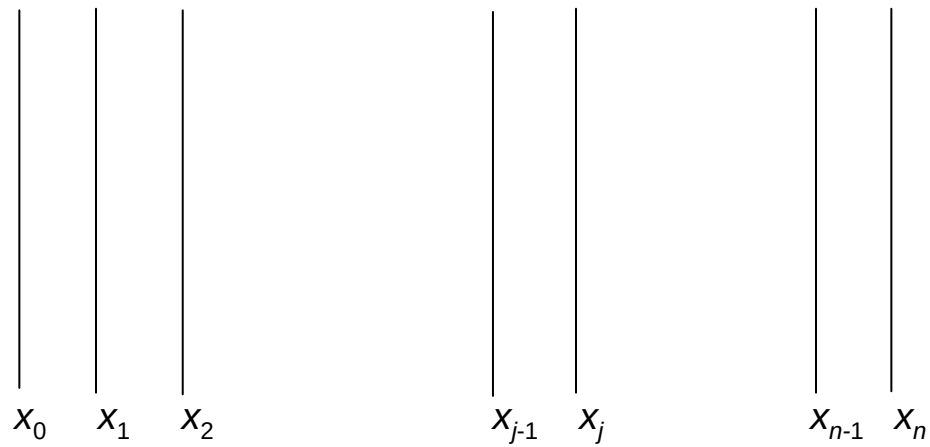
+ Is there a more concise, more theoretically friendly analytical approach to heterogeneous MG diffusion theory?

Consequences:

+ Theory applies equally regardless of the number of regions and groups
+ Educational enrichment: New solution to an old problem
+ Nodal method application (2D/3D)
+ Transient applications
+ Curvilinear geometries
+ n-γ application

I. **Theory:**

1. Fundamental Assumptions:
   - a. Heterogeneous medium
   - b. Fission neutrons to all groups
   - c. Fission occurs in all groups
   - d. Up/Down scatter to all groups (full stride)
   - e. Steady state spatially dependent source



Heterogeneous Medium

2. Governing Steady State Diffusion equation for homogeneous region *j*

$$\left[D_{gj}\frac{d^2}{dx^2}-\Sigma_{gj}\right]\phi_{gj}(x)+\chi_g\sum_{g'=1}^{G}\nu\Sigma_{fg'j}\phi_{g'j}(x)+ \qquad 1\le j\le n-1,\quad 1\le g\le G$$

$$+\sum_{g'=1}^{G}\Sigma_{gg'j}\phi_{g'j}(x)=-Q_{gj}(x)$$

a. In vector form for region *j*: $\boldsymbol{M}_{j,G}(x)\boldsymbol{\phi}_j(x)=-\boldsymbol{q}_j(x)$

$$\boldsymbol{M}_{j,G}(x)\equiv\begin{bmatrix} d^2/dx^2+\gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots\cdots & \gamma_{1G} \\ \gamma_{21} & d^2/dx^2+\gamma_{22} & \gamma_{23} & \cdots\cdots & \gamma_{2G} \\ \gamma_{31} & \gamma_{32} & d^2/dx^2+\gamma_{33} & \cdots\cdots & \gamma_{3G} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \gamma_{G1} & \gamma_{G2} & \cdots\cdots & \cdots\cdots & d^2/dx^2+\gamma_{GG} \end{bmatrix} \qquad \boldsymbol{\phi}(x)\equiv\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\\ \phi_3(x)\\ \cdots\\ \phi_G(x)\end{bmatrix} \qquad \boldsymbol{q}(x)\equiv\begin{bmatrix}Q_1(x)/D_1\\ Q_2(x)/D_2\\ Q_3(x)/D_3\\ \cdots\\ Q_G(x)/D_G\end{bmatrix}$$

+ For $g=1,2,\ldots G,\qquad g'=1,2,\ldots G$
the *g*-matrix is defined in each
region as (regional subscript is suppressed)

$$\gamma_{gg}\equiv\frac{\chi_g\nu\Sigma_{fg}-\left(\Sigma_g-\Sigma_{gg}\right)}{D_g} \qquad \gamma_{gg'}\equiv\frac{\chi_g\nu\Sigma_{fg'}+\Sigma_{g'g}}{D_g},\quad g\ne g'$$

3. General solution

$$\boldsymbol{\phi}_j(x) = \boldsymbol{\Psi}_j(x) + \boldsymbol{\phi}_{p,j}(x)$$

$\boldsymbol{\Psi}_j(x) \equiv$ Homogeneous solution

$\boldsymbol{\phi}_{p,j}(x) \equiv$ Particular solution

Straightforward Solution Strategy:

(A) Solve the following without regard to BCs

$$\boldsymbol{M}_{j,G}(x)\boldsymbol{\Psi}_j(x) = 0$$

$$\boldsymbol{M}_{j,G}(x)\boldsymbol{\phi}_{p,j}(x) = -\boldsymbol{q}_j(x)$$

(B) Apply BCs to general solution above

4. First consider the homogeneous solution for $\boldsymbol{M}_{j,G}(x)\boldsymbol{\Psi}_j(x)=\mathbf{0}$

a. Seek solution in terms of eigenvalues B_j^2

$$\left[\nabla^2 + B_j^2\boldsymbol{I}\right]\boldsymbol{\Psi}_j(x)=\mathbf{0}$$

$$\nabla^2 \equiv \begin{bmatrix} \frac{d^2}{dx^2} & 0 & .... & .... & 0 \\ 0 & \frac{d^2}{dx^2} & 0 & .... & 0 \\ 0 & .... & .... & ... & 0 \\ .... & .... & .... & .... & .... \\ 0 & .... & .... & 0 & \frac{d^2}{dx^2} \end{bmatrix} = \frac{d^2}{dx^2}\boldsymbol{I}$$

b. Original operator becomes

$$\boldsymbol{M}_{j,G}(x) \equiv \frac{d^2}{dx^2}\boldsymbol{I} + \boldsymbol{\gamma}_j$$

$$\boldsymbol{\gamma}_j \equiv \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & .... & \gamma_{1,G} \\ \gamma_{2,1} & ..... & & ..... \\ ..... & & & \\ & & & ..... \\ \gamma_{G,1} & ..... & & \gamma_{G,G} \end{bmatrix}$$

$$\boldsymbol{M}_{j,G}\left(B_j^2\right)\boldsymbol{\Psi}_j(x)=\mathbf{0}$$

$$\boldsymbol{M}_{j,G}\left(B_j^2\right) \equiv \begin{bmatrix} \gamma_{1,1}-B_j^2 & \gamma_{1,2} & .... & \gamma_{1,G} \\ \gamma_{2,1} & ..... & & ..... \\ ..... & & & \\ & & & ..... \\ \gamma_{G,1} & ..... & & \gamma_{G,G}-B_j^2 \end{bmatrix}$$

$$\therefore\ Det\left[\boldsymbol{M}_{j,G}\left(B_j^2\right)\right]=0 \quad \Rightarrow B_{jk}^2,\ k=1,2,\ldots G$$

+ Note: $B_{j,k}^2$ are assumed to be distinct but may be complex in conjugate pairs

c. For each *k-mode* therefore $\left[\frac{d^2}{dx^2}+B_{jk}^2\right]\Psi_{j,k}(x)=0$

+ Each (group) component will have two independent solutions

$$\Psi_{jgk}(x)=C_{jgk}^{+}h_{jk}^{+}(x)+C_{jgk}^{-}h_{jk}^{-}(x) \qquad \left[\frac{d^2}{dx^2}+B_{jk}^2\right]h_{jk}^{\pm}(x)=0$$

Note: Since B_{jk}^2 can be complex, the coefficients $C_{jgk}^{\pm}$ can also be complex

d. General solution representation for group g is a sum of all possible solution modes

$$\Psi_{gj}(x)=\sum_{k=1}^{G}\left[C_{jgk}^{+}h_{jk}^{+}(x)+C_{jgk}^{-}h_{jk}^{-}(x)\right]$$

$\Psi_{jk}(x)$ must be real

+ To solve: $\left[\frac{d^2}{dx^2}+B_{jk}^2\right]h_{jk}^{\pm}(x)=0$

subject to the convenient conditions

$$h_{jk}^{+}(x_j)\equiv 1 \qquad h_{jk}^{-}(x_j)\equiv 0$$

$$h_{jk}^{+}(x_{j-1})\equiv 0 \qquad h_{jk}^{-}(x_{j-1})\equiv 1$$

Considering a 2D set of basis solutions with above conditions

$$\begin{cases} h_{jk}^{+}(x)=\left[\dfrac{\sin\left(B_{jk}\left(x-x_{j-1}\right)\right)}{\sin\left(B_{jk}\Delta_j\right)}\right] \\ h_{jk}^{-}(x)=\left[\dfrac{\sin\left(B_{jk}\left(x_j-x\right)\right)}{\sin\left(B_{jk}\Delta_j\right)}\right] \end{cases}$$

Note: The argument of the sine can be complex

+ Representation of $C^{\pm}_{jgk}$

From original equations by group written as

$$\frac{d^2\Psi_{gj}(x)}{dx^2}+\sum_{g'=1}^{G}\gamma_{gg'}\Psi_{g'j}(x)=0$$

and the general solution by group-- implies

$$\Rightarrow B^2_{jk}C^{\pm}_{jgk}-\sum_{g'=1}^{G}\gamma_{gg'}C^{\pm}_{jg'k}=0,\quad k=1,2,\ldots G$$

Set of homogeneous equations of rank *G*-1 for each *k*. Thus, there is a one-parameter family of solutions that can be expressed in terms of an arbitrary constant. We choose that constant such that

$$C^{\pm}_{jgk}=\alpha_{gk}C^{\pm}_{j1k},\ g=2,3,\ldots G$$

For consistency: $\alpha_{1k}\equiv 1,\ k=1,2,\ldots G$

$$\Rightarrow \sum_{g'=2}^{G}\left[B^2_{jk}\delta_{gg'}-\gamma_{gg'}\right]\alpha_{g'k}=\gamma_{g1},\quad g=2,3,\ldots G,\ k=1,2,\ldots G$$

e. Intermediate expression for homogeneous solution:

$$\Psi_{gj}(x)=\sum_{k=1}^{G}\left[\alpha_{gk}h_{jk}^{+}(x)C_{j1k}^{+}+\alpha_{gk}h_{jk}^{-}(x)C_{j1k}^{-}\right]$$

or in vector form

$$\boldsymbol{\Psi}_{j}(x)=\boldsymbol{\alpha}_{j}\boldsymbol{h}_{j}^{+}(x)\boldsymbol{C}_{j1}^{+}+\boldsymbol{\alpha}_{j}\boldsymbol{h}_{j}^{-}(x)\boldsymbol{C}_{j1}^{-}$$

$$\boldsymbol{\alpha}_{j}\equiv\begin{bmatrix}1 & 1 & \dots & \dots & 1\\ \alpha_{2,1}^{j} & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ \alpha_{G,1}^{j} & \dots & \dots & \dots & \alpha_{G,G}^{j}\end{bmatrix}\qquad \boldsymbol{h}_{j}^{\pm}(x)\equiv\begin{bmatrix}h_{j1}^{\pm}(x) & 0 & \dots & 0\\ 0 & h_{j2}^{\pm}(x) & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & \dots & \dots & h_{jG}^{\pm}(x)\end{bmatrix}\qquad \boldsymbol{C}_{j1}^{\pm}\equiv\begin{bmatrix}C_{j11}^{\pm}\\ C_{j12}^{\pm}\\ \dots\\ C_{j1G}^{\pm}\end{bmatrix}$$

f. Determination of $\boldsymbol{C}_{j1}^{\pm}$

+ The general solution is of the form

$$\boldsymbol{\phi}_j(x) = \boldsymbol{\alpha}_j \boldsymbol{h}_j^+(x)\boldsymbol{C}_{j1}^+ + \boldsymbol{\alpha}_j \boldsymbol{h}_j^-(x)\boldsymbol{C}_{j1}^- + \boldsymbol{\phi}_{p,j}(x)$$

+ If we define $\boldsymbol{\phi}_j \equiv \boldsymbol{\phi}_j(x_j)$

$\boldsymbol{\phi}_{j-1} \equiv \boldsymbol{\phi}_{j-1}(x_{j-1}) = \boldsymbol{\phi}_j(x_{j-1})$ (from flux continuity)

then at $x = x_j$ and $x = x_{j-1}$

$$\boldsymbol{\phi}_j(x_j) - \boldsymbol{\phi}_{p,j}(x_j) = \boldsymbol{\alpha}_j \cancelto{\boldsymbol{I}}{\boldsymbol{h}_j^+(x_j)}\boldsymbol{C}_{j1}^+ + \boldsymbol{\alpha}_j \cancelto{\boldsymbol{0}}{\boldsymbol{h}_j^-(x_j)}\boldsymbol{C}_{j1}^-$$

$$\boldsymbol{\phi}_j(x_{j-1}) - \boldsymbol{\phi}_{p,j}(x_{j-1}) = \boldsymbol{\alpha}_j \cancelto{\boldsymbol{0}}{\boldsymbol{h}_j^+(x_{j-1})}\boldsymbol{C}_{j1}^+ + \boldsymbol{\alpha}_j \cancelto{\boldsymbol{I}}{\boldsymbol{h}_j^-(x_{j-1})}\boldsymbol{C}_{j1}^-$$

giving

$$\boldsymbol{C}_{j1}^+ = \alpha_j^{-1}\left[\boldsymbol{\phi}_j - \boldsymbol{\phi}_{pj}^+\right] \qquad \boldsymbol{\phi}_{pj}^+ \equiv \boldsymbol{\phi}_{p,j}(x_j)$$

$$\boldsymbol{C}_{j1}^- = \alpha_j^{-1}\left[\boldsymbol{\phi}_{j-1} - \boldsymbol{\phi}_{pj}^-\right] \qquad \boldsymbol{\phi}_{pj}^- \equiv \boldsymbol{\phi}_{p,j}(x_{j-1})$$

+ The final form of the solution then becomes

$$\boldsymbol{\phi}_j(x) = \left[\boldsymbol{\alpha}_j \boldsymbol{h}_j^+(x) \boldsymbol{\alpha}_j^{-1}\right]\left(\boldsymbol{\phi}_j - \boldsymbol{\phi}_{p,j}^+\right) + \left[\boldsymbol{\alpha}_j \boldsymbol{h}_j^-(x) \boldsymbol{\alpha}_j^{-1}\right]\left(\boldsymbol{\phi}_{j-1} - \boldsymbol{\phi}_{p,j}^-\right) + \boldsymbol{\phi}_{p,j}(x)$$

- Now same form as the one-group case
- For convenience let

$$\boldsymbol{A}_j(x) \equiv \boldsymbol{\alpha}_j \boldsymbol{h}_j^+(x) \boldsymbol{\alpha}_j^{-1}$$
$$\boldsymbol{B}_j(x) \equiv \boldsymbol{\alpha}_j \boldsymbol{h}_j^-(x) \boldsymbol{\alpha}_j^{-1}$$

Need to be real

$$\boldsymbol{\phi}_j(x) = \boldsymbol{A}_j(x)\left(\boldsymbol{\phi}_j - \boldsymbol{\phi}_{pj}^+\right) + \boldsymbol{B}_j(x)\left(\boldsymbol{\phi}_{j-1} - \boldsymbol{\phi}_{pj}^-\right) + \boldsymbol{\phi}_{p,j}(x)$$

Note: $\boldsymbol{\phi}_j$ are not known

5. Current continuity and recurrence

a. Current continuity at interfaces requires

$$-\boldsymbol{D}_{j-1}\left.\frac{d\boldsymbol{\phi}_{j-1}(x)}{dx}\right|_{x_{j-1}} = -\boldsymbol{D}_j\left.\frac{d\boldsymbol{\phi}_j(x)}{dx}\right|_{x_j}, \quad 2 \le j \le n$$

gives

$$\boldsymbol{M}_j\boldsymbol{\phi}_j - \boldsymbol{N}_j\boldsymbol{\phi}_{j-1} - \boldsymbol{P}_j\boldsymbol{\phi}_{j-2} = \boldsymbol{f}_j, \quad 2 \le j \le n$$

$$\boldsymbol{M}_j \equiv \boldsymbol{D}_j\left.\frac{d\boldsymbol{A}_j(x)}{dx}\right|_{x_{j-1}}$$

$$\boldsymbol{N}_j \equiv \boldsymbol{D}_{j-1}\left.\frac{d\boldsymbol{A}_{j-1}(x)}{dx}\right|_{x_{j-1}} - \boldsymbol{D}_j\left.\frac{d\boldsymbol{B}_j(x)}{dx}\right|_{x_{j-1}}$$

$$\boldsymbol{P}_j \equiv \boldsymbol{D}_{j-1}\left.\frac{d\boldsymbol{B}_{j-1}(x)}{dx}\right|_{x_{j-1}}$$

$$\boldsymbol{f}_j \equiv \boldsymbol{D}_j\left[\left.\frac{d\boldsymbol{A}_j(x)}{dx}\right|_{x_{j-1}}\boldsymbol{\phi}_{pj}^{+} + \left.\frac{d\boldsymbol{B}_j(x)}{dx}\right|_{x_{j-1}}\boldsymbol{\phi}_{p,j}^{-} - \left.\frac{d\boldsymbol{\phi}_{p,j}(x)}{dx}\right|_{x_{j-1}}\right] -$$

$$-\boldsymbol{D}_{j-1}\left[\left.\frac{d\boldsymbol{A}_{j-1}(x)}{dx}\right|_{x_{j-1}}\boldsymbol{\phi}_{pj-1}^{+} + \left.\frac{d\boldsymbol{B}_{j-1}(x)}{dx}\right|_{x_{j-1}}\boldsymbol{\phi}_{p,j-1}^{-} - \left.\frac{d\boldsymbol{\phi}_{p,j\text{-}1}(x)}{dx}\right|_{x_{j-1}}\right]$$

6. Free surface boundary conditions: Closure

a. Zero flux

$$\phi_0 = \phi_n = \mathbf{0}$$

b. Zero current:

$$\left.\frac{d\phi_0(x)}{dx}\right|_{x_0} = \left.\frac{d\phi_n(x)}{dx}\right|_{x_n} = \mathbf{0}$$

+ Can reformulate conditions on ϕ_0 and ϕ_n such that

$$\phi_0 = \phi_n = 0$$

c. To solve:

$$\boldsymbol{M}_j\phi_j - \boldsymbol{N}_j\phi_{j-1} - \boldsymbol{P}_j\phi_{j-2} = \boldsymbol{f}_j, \quad 2 \le j \le n$$

with $\phi_0 = \phi_n = 0$

7. Solution to recurrence relation (primarily for criticality)

$$\boldsymbol{M}_j\boldsymbol{\phi}_j - \boldsymbol{N}_j\boldsymbol{\phi}_{j-1} - \boldsymbol{P}_j\boldsymbol{\phi}_{j-2} = \boldsymbol{f}_j, \quad 2 \le j \le n$$

a. Assume form of solution is (for general condition at j = 0)

$$\boldsymbol{\phi}_j = \boldsymbol{g}_j\boldsymbol{\phi}_0 + \rho_j\boldsymbol{\phi}_1 + \sum_{l=2}^{j}\boldsymbol{\beta}_{j,l}\boldsymbol{f}_l$$

where the complementary solutions are

$$\boldsymbol{M}_j\begin{bmatrix}\boldsymbol{g}_j\\ \rho_j\end{bmatrix} - \boldsymbol{N}_j\begin{bmatrix}\boldsymbol{g}_{j-1}\\ \rho_{j-1}\end{bmatrix} - \boldsymbol{P}_j\begin{bmatrix}\boldsymbol{g}_{j-2}\\ \rho_{j-2}\end{bmatrix} = \boldsymbol{0}$$

$$\begin{bmatrix}\boldsymbol{g}_0\\ \rho_0\end{bmatrix} \equiv \begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix} \qquad \begin{bmatrix}\boldsymbol{g}_1\\ \rho_1\end{bmatrix} \equiv \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix}$$

b. Satisfaction of BC at $x = x_n$

$$\boldsymbol{\phi}_1 = -\rho_n^{-1}\left[\boldsymbol{g}_n\boldsymbol{\phi}_0 + \sum_{l=2}^{n}\boldsymbol{\beta}_{j,l}\boldsymbol{f}_l\right]$$

$$\boldsymbol{\phi}_j = \left[\boldsymbol{g}_j - \rho_n^{-1}\rho_j\boldsymbol{g}_n\right]\boldsymbol{\phi}_0 - \rho_j\,\rho_n^{-1}\sum_{l=2}^{n}\boldsymbol{\beta}_{j,l}\boldsymbol{f}_l + \sum_{l=2}^{j}\boldsymbol{\beta}_{j,l}\boldsymbol{f}_l$$

c. Criticality:

+ Obvious *n* by *G* determinant condition

$$\mathrm{Det}\begin{bmatrix} -N_1 & M_1 & 0 & \dots & & 0 \\ 0 & -P_2 & -N_2 & M_2 & & \dots \\ \dots & 0 & \dots & \dots & \dots & \\ & & \dots & \dots & 0 & \dots \\ \dots & & -P_{n-1} & -N_{n-1} & M_{n-1} & 0 \\ 0 & \dots & & 0 & -P_n & -N_n \end{bmatrix} = 0$$

+ New criticality condition:

Require $\boldsymbol{f}_l$ and $\boldsymbol{\phi}_0$ to vanish

Require any interface flux vector to be nonzero

- In particular for *j* = 1

$$\boldsymbol{\rho}_n \boldsymbol{\phi}_1 = -\left[\boldsymbol{g}_n \boldsymbol{\phi}_0 + \sum_{l=2}^{n} \boldsymbol{\beta}_{j,l} \boldsymbol{f}_l \right] = 0$$

- Implies $\boldsymbol{\rho}_n$ must be singular $\Rightarrow Det\left[\boldsymbol{\rho}_n\left(k_{eff}\right)\right] \equiv 0$

(like the denominator for the one-group case)

8. Explicit solution to recurrence relation

+ Consider: $\boldsymbol{M}_j\boldsymbol{\rho}_j - \boldsymbol{N}_j\boldsymbol{\rho}_{j-1} - \boldsymbol{P}_j\boldsymbol{\rho}_{j-2} = \mathbf{0}$

$$\boldsymbol{\rho}_{j-1} - \boldsymbol{\rho}_{j-1} = \mathbf{0}$$

+ Reformulate with $\boldsymbol{y}_j \equiv \begin{bmatrix} \boldsymbol{\rho}_j \\ \boldsymbol{\rho}_{j-1} \end{bmatrix}$

$$\Rightarrow \boldsymbol{A}_j\boldsymbol{y}_j - \boldsymbol{B}_j\boldsymbol{y}_{j\text{-}1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{A}_j \equiv \begin{bmatrix} \boldsymbol{M}_j & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \qquad \boldsymbol{B}_j \equiv \begin{bmatrix} \boldsymbol{N}_j & \boldsymbol{P}_j \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix} \qquad \rho_0 = 0 \quad \boldsymbol{\rho}_1 = \boldsymbol{I} \qquad \Rightarrow \boldsymbol{y}_1 \equiv \begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix}$$

$$\left.\begin{aligned} &\boldsymbol{y}_j - \boldsymbol{A}_j^{\text{-}1}\boldsymbol{B}_j\boldsymbol{y}_{j\text{-}1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \\ &\boldsymbol{G}_j \equiv \boldsymbol{A}_j^{\text{-}1}\boldsymbol{B}_j \\ &\boldsymbol{y}_j = \begin{bmatrix} \boldsymbol{\rho}_j \\ \boldsymbol{\rho}_{j-1} \end{bmatrix} = \left[\prod_{l=2}^{j}\boldsymbol{G}_l\right]\boldsymbol{y}_1 = \left[\prod_{l=2}^{j}\boldsymbol{G}_l\right]\begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix} \end{aligned}\right\} \text{-- implies } \boldsymbol{\rho}_j = \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \end{bmatrix}\left[\prod_{l=2}^{j}\boldsymbol{G}_l\right]\begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix}$$

+ Criticality: $Det[\boldsymbol{\rho}_n] = Det\left\{\begin{bmatrix} \boldsymbol{I} & \mathbf{0} \end{bmatrix}\left[\prod_{l=2}^{n}\boldsymbol{G}_l\right]\begin{bmatrix} \boldsymbol{I} \\ \mathbf{0} \end{bmatrix}\right\}$

9. The particular solution

$$\left[\nabla^2+\gamma_j\right]\phi_{p,j}(x)=-\boldsymbol{q}_j(x)$$

+ Apply variation of parameters

$$\phi_{p,j}(x)=\boldsymbol{\alpha}_j\boldsymbol{h}_j^+(x)\boldsymbol{u}_1(x)+\boldsymbol{\alpha}_j\boldsymbol{h}_j^-(x)\boldsymbol{u}_2(x)$$

to find

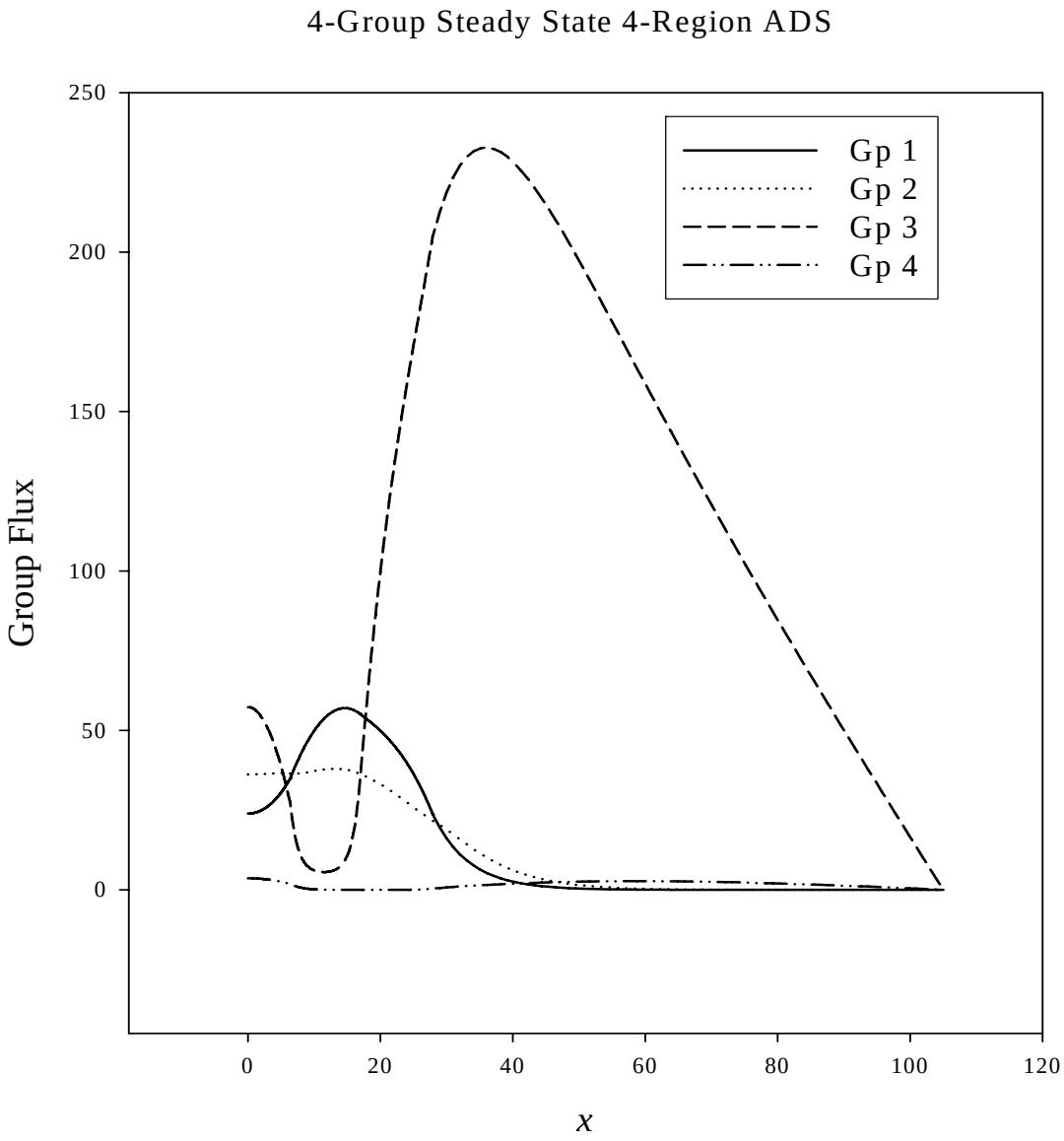
$$\phi_{p,j}(x)=\boldsymbol{\alpha}_j\boldsymbol{W}^{-1}\boldsymbol{h}_j^+(x)\int_x^{x_j}dx\boldsymbol{h}_j^-(x)\boldsymbol{\alpha}_j^{-1}\boldsymbol{q}_j(x)+$$

$$+\boldsymbol{\alpha}_j\boldsymbol{W}^{-1}\boldsymbol{h}_j^-(x)\int_{x_{j-1}}^{x}dx\boldsymbol{h}_j^+(x)\boldsymbol{\alpha}_j^{-1}\boldsymbol{q}_j(x)$$

$$\boldsymbol{W}\equiv\mathrm{diag}\left\{\frac{B_{jk}}{\sin\left(B_{jk}\Delta_j\right)}\right\}$$

II Some Results

a. ADS steady State for 4-group xsecs of 10 September 2003 for uniform source in first region

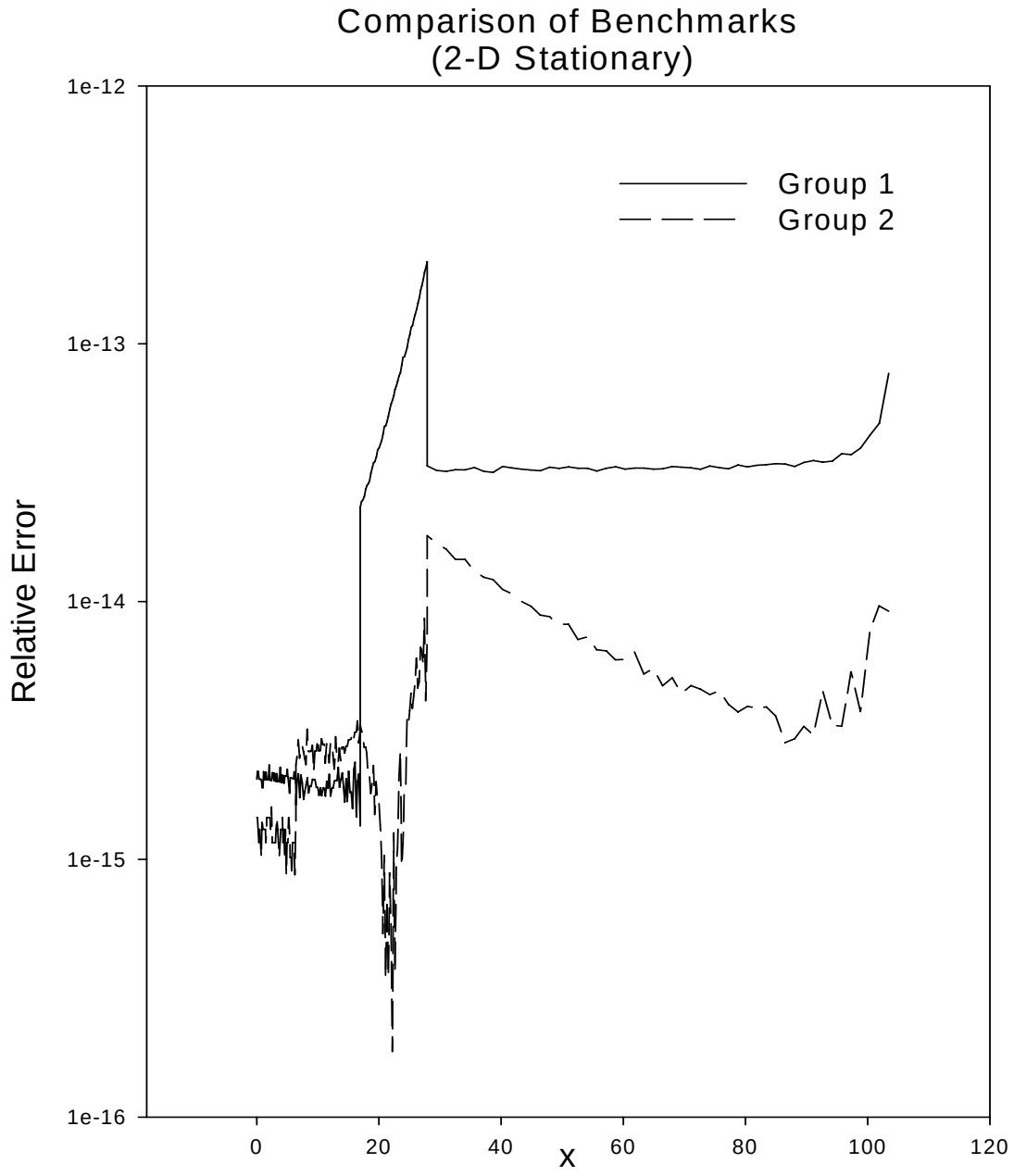


- By the *n* by *G* determinant

Mode	***k_{eff}***
Double Precision	9.8825122560**660** E-01
Quad Precision	9.8825122560341 E-01
Mathematica	9.88251225603411E-01

- By the *G* by *G* determinant but also a recurrence for ρ_n

Mode	*k_{eff}*
Double Precision	9.8825122**857172** E-01
Quad Precision	9.8825122560341 E-01
Mathematica	9.88251225603411E-01


Comparison of Benchmarks
(2-D Stationary)
Group 1
Group 2
Relative Error
1e-12
1e-13
1e-14
1e-15
1e-16
0
20
40
60
80
100
120
x

III Some Discussion and Comments

+ Curvilinear
+ Basis for 2D/3D Nodal algorithm
+ Transients via LTI
+ Other?